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SAFETY AND THE PROBABILITY OF STRUCTURAL FAILURE

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SAFETY AND THE PROBABILITY OF STRUCTURAL FAILURE

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INTRODUCTION

The preoccupation, in recent years, on the part of structural engineers with problems of the carrying capacity and of the safety of structures is the sign of a growing realization that significant advances in the theory of structures can be expected from a critical re-evaluation of the basis and the premises of such theory rather than from further refinement of the methods of stress-analysis. Present methods of structural analysis produce designs the safety of which is neither balanced nor clearly specified; the formulation of a rational concept of safety and of methods for the evaluation of a measure of safety represent tasks of considerable urgency in structural research. The fact that current design procedures in the majority of cases result, fortunately, in structures of excessive though unbalanced safety rather than in unsafe structures is an indication of the caution exercised by designers and the authorities responsible for design specifications rather than of the reliability and accuracy of design procedures.

From the realization that the design, at present, of a structure for future use must necessarily embody predictions of expected performance of the structural material based on past experience (materials tests), as well as predictions of expected load-patterns and intensities extrapolated from past observations (load records), it follows, that since such predictions by their very nature cannot be made with certainty but only with a certain measure of probability, the concept of probability must form an integral part of any rational design: any conceivable condition of the designed structure is necessarily associated with a numerical measure of the probability of its occurrence. It is by this measure alone that the reality and structural significance of a specified condition can be evaluated.

This does not imply that the use of probability theory is in itself sufficient to make design procedures more adequate and reliable; probability concepts and statistical methods based thereupon, while extremely valuable tools in design, can be effectively used only in conjunction with a thorough knowledge of the operating conditions of the structure and of its structural action, and with sound engineering judgment. They will provide objective guidance for such judgment whenever random phenomena and random magnitudes have to be considered; they are useless in the analysis of phenomena and effects the variation of which is not random.

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The ignorance of the causes of variation of a phenomenon does not in itself make such variations random. To produce randomness it is necessary that the variability be controlled by a large number of contributing causes, the effects of any one of which is small in comparison to that of all the others; the identification of the effect of any contributing cause, whether known or unknown, in the resulting variation becomes, therefore, either impossible or irrelevant. Once randomness is established, however, the rules governing variation are as rigid as those governing the relations between cause and effect in classical mechanics.

I. Design Procedure

1- Conditions of Failure and of "Unserviceability"

The principal aspects of the operating conditions of a load-carrying structure are the limiting conditions of service and of failure. Both can be rationally specified only in terms of a characteristic load carried under the defined conditions. Specification of the critical operating conditions of a structure in terms of stresses rather than loads which forms the basis of current design procedures, presuppose proportionality between the loads carried by and the stresses induced in the structure. The validity of this assumption is, however, limited to perfectly elastic structures, the deformations of which are small in relation to their dimensions. Since real structures under practically all loading conditions fail after more or less extensive inelastic deformation accompanied by substantial modifications of the initial (elastic) structural action, the specification of the failure condition in terms of a stress cannot reflect the behavior of the structure at or close to the instant of failure.

Under conditions of service the structure should be able to sustain an infinite number or, in certain cases, a specified finite number of load applications without any noticeable effect on its usefulness or "serviceability". Since, prior to failure, this usefulness can only be reduced by such a change of shape, permanent or transient, which would impair the function of the structure, the specification of service conditions should be based on the requirement of perfect reversibility of the deformation of the structure in conjunction with a limitation of the maximum value of such deformation. The limiting condition of "serviceability" is therefore to be defined by a characteristic or standard load the repeated application of which would produce reversible deformation of the limiting magnitude.

This definition in terms of reversibility does not imply a perfectly elastic condition of the structure at all times. The most important condition for the successful design of structures on the basis of elastic stress analysis is the existence of an inelastic response of the structural material, sufficiently pronounced to reduce to manageable levels the unavoidable stress-concentrations in and close to structural connections, points of load application and changes in cross-section; in a perfectly elastic structure these concentrations would cause fracture under loads far below any reasonable service load. As a result of

localized inelastic strains produced in the vicinity of the stress-concentration by the first few load applications it is therefore possible to create residual stresses which under subsequent applications of smaller loads, prevent further inelastic deformation of the structure. This load, therefore, qualifies as a "service load" in spite of the permanent deformation produced by its first few applications, provided such initial permanent deformation does not impede the effective functioning of the structure.

There appears to be no valid reason for restricting the definition of limiting "service load" to loads which, on initial application, produce limited permanent deformation only in the connections and points of load application. Provided the permanent deformation resulting from initial loading is rapidly stabilized and its magnitude is not objectionable from the point of view of the function of the structure the specification of "serviceability" can be extended to such loads which, in the first few applications, cause a permanent deformation of acceptable magnitude within certain structural elements, followed by fully reversible deformations under all subsequent applications of the same or smaller loads. Such a specification of the maximum service load has, in fact, been established in the design of gun-barrels ("auto-frettage"), steel containers for high internal pressures, heavy coil springs, crane-hooks and other machine parts with highly non-uniform elastic stress destribution; it cannot be easily translated into conventional terms of stress.

2. Conditions of Loading

The "load" in terms of which unserviceability and failure of a structure are to be defined is a relatively simple force pattern that is or can be made representative of the variety of actual loads to which the structure might be subjected during its period of service. It is expedient to distinguish between "simple" or "proportional" loads that do not vary in location, distribution, or composition but only in intensity, and "complex" loads, which vary in location, distribution, composition as well as in intensity. Depending on the range of variation in location and distribution certain types of "complex" loads can, in fair approximation, be considered as "proportional".

True proportional loads are, for instance, acceleration and landing forces acting on airplanes, seismic forces, column loads, the weight of fluids in storage tanks, as well as forces resulting from restriction imposed on the deformation of the structure, such as temperature forces; "complex" loads that can be approximated by "proportional" loads are traffic loads on simple single span bridge-members which can accommodate not more than a single traffic unit, floor loads of buildings, as well as wind-, gust-, and blast-forces. "Complex" loads that cannot be reduced to "simple" loads are movable forces the variation in location of which may produce a change in sign of the resulting effect, as well as load-combinations the components of which are independently variable, such as the combination of bending moments due to wing-loads (depending on acceleration,) and torque (depending on speed) which jointly act on aircraft wings, or axial loads and bending moments acting jointly on frame-columns. The "complexity" of the load is the result of the type of the structure not less than of the load. In general, design

procedures are much simpler for proportional loads than for complex loads.

The definition of the critical operating conditions requires the specification of a characteristic load pattern, of the expected range of variation of the load intensity, of the expected range of variation of the rates of load application, of the duration of loading cycles, as well as of the characteristic pattern of the expected load-sequences. The response of the designed structure, in terms of deformation and failure, may change considerably with the variation of any one of these characteristics.

3. Sequence of Design

A rational procedure of structural design, in which the aspects of operating conditions and structural action are integrated with that of the probability of their occurrence, should embody the following steps:

A. Load Analysis:

1) Specification of the standard load pattern, and of its relation to actual loading conditions;
2) Evaluation of the probability of occurrence of various intensities of the standard load;
3) Analysis of the time-dependence of the standard load and of its variability.

B. Structural Analysis:

4) Determination of the maximum intensity of the standard load which can be applied to the idealized structure without causing structural failure during the expected period of service, either on a single application or after repeated or sustained application. The idealization refers to the deformational response of the structural material, as the structure approaches the failure load.
5) Evaluation of the probability distribution of the standard-load-intensity producing failure, resulting from random variations of the deformation and strength properties of the real material, and from the effect of random variations of loading rate and of duration and sequence of load applications on those properties.
6) Determination of the limiting intensity of the standard load, the repeated application of which does not impair the "serviceability" of the idealized structure, as defined by functionally acceptable limits of the permanent deformation under the first applications of the load, and of the recoverable deformation under all subsequent load applications.
7) Evaluation of the probability distribution of the limiting intensity of the standard load of "unserviceability", resulting from random variation of the deformational properties of the real material and of their variation due to random variation of rate, duration and sequence of loading.

C. Safety Analysis:

8) Formulation of "decision rules" for the selection of the design conditions of the structure with respect to unserviceability and failure, and choice of the optimal rule. These rules can be formulated on the basis of two fundamentally different approaches:
   a) by the direct probability approach, specifying numerical values for the probabilities with which both failure and unserviceability are to be avoided,
   b) by the economic approach, in which the cost of increasing the carrying capacity of the structure is weighed against the risk and eventual cost of failure, specifying that the sum of the cost of the structure and the probable cost of failure or of unserviceability shall be a minimum.

9) Determination of the intensities of the standard load associated with the acceptable probabilities of failure and of unserviceability, that have resulted from the selected decision rules. The lower of these intensities, which represents the limiting standard load for which the structure is designed with the specified probability of either failure or unserviceability is then related to the real loading conditions by inversion of step (1) of the outlined design procedure, thus completing the design.

4. The Concept of Safety

A procedure of structural design can thus be established in which the conventional engineering concepts of safety and safety factors, load factors or permissible stresses do not appear at all. Considering the individual steps of the procedure outlined above it is, in fact, not immediately obvious how any of these concepts could be introduced without disturbing, by arbitrary assumptions, the sequence of design for a specified probability of failure or unserviceability.

However, since the concept of safety is deeply rooted in engineering design practice, while the notion that a finite, no matter how small, probability of failure, or at least of unserviceability is necessarily associated with every reasonably efficient design is repulsive to a majority of engineers, it appears desirable to retain the design concept of "safety", rather than replace it by that of "probability of failure", but to reformulate the former in terms of the latter. The safety factor is thus transformed into a parameter that is a function of the random variation of all design characteristics, as well as of the non-random variations essentially due to the process of construction. Only the part representing the random variations can be derived from the relevant probabilities and represents, therefore, the objective minimum value of the safety factor. The selection of the actual engineering safety factor should be guided by this minimum value, in conjunction with sound engineering judgment concerning the possible uncertainties associated with actual construction.

In a previous paper[1] an attempt has been made to derive this minimum safety factor from the statistical variations of the design

characteristics under the simplest possible assumption concerning structural action, namely, invariability of such action over the entire range of operating conditions, including the limiting condition of failure. In this case the minimum safety factor and the associated probability of failure are obtained by correlating the stress in excess of the design stress produced by a rather improbably high load fluctuation, with the difference between the design strength and an improbably low value of strength of the structural material, and by multiplying the probabilities of occurrence of those extreme values of stress and strength.

Since the publication of this and a subsequent paper[2] the concept of safety has been extensively discussed in various countries; particularly in England[3], France[4], Spain[5], and Sweden[6]. The subject of the safety of structures has also been placed on the agenda of the last two International Congresses for Bridge and Structural Engineering in 1948[7] and 1952[8] as one of the main topics of discussion. In this country and in England committees have been set up by the ASCE and the Inst. of Structural Engineers, respectively, to deal with the subject of structural safety, and to recommend such modifications of standard structural design procedures which would embody the most adequate formulation of the concept of safety. Considerable effort has also been made in recent years in this country[9] and in England[10] to evaluate failure-loads of structures of idealized inelastic response, particularly of ideal plastic structures; part of this work represents however, duplication of much earlier research in the same direction[11]. Finally, investigations have been made or are in progress which attempt to determine the actual shape of the statistical variation of various types of loadings, such as highway loads[12], building-floor-loads[6] wind- and gust-forces[13], as well as of the deformational and strength properties of structural materials, such as the yield-point of structural steel[14], the compressive strength of concrete[15], the number of stress cycles to fatigue fracture (fatigue life)[16], the fatigue strength[17], and the energy absorbed in dynamic fracture under specific testing conditions[18].

The ultimate purpose of each one of the lines of investigation referred to above is to contribute to the improvement of the procedures of structural design. Hence, their relative importance can be judged in terms of their relation to the individual steps of the previously outlined design sequence. On the basis of this interrelation it becomes obvious that no single line of investigation alone can produce a solution of the design problem; this fact illustrates the futility of the claim, not infrequently heard, that the application of plastic or "limit" design methods represents the complete rational solution of the design problem. In itself the "plastic" concept of safety is as limited as the conventional one; only an integrated approach to the design problem as a whole, considering all steps in the design procedure, can lead to such a solution.

II. Load Analysis

5. The Standard Load

Since the analysis of a structure cannot be better than the load-analysis underlying it, refinement of structural analysis cannot make up for

crudeness of load analysis. In spite of its self-evidence, this statement is frequently forgotten in structural design.

Whenever the operating conditions of the structures are such that the loads are of different patterns and of different weights as, for instance, in highway bridges carrying different vehicle types and loads, a "standard load", has to be established. This standard load defines a loading pattern which is sufficiently characteristic of the real load patterns to make the structural action equivalent or comparable; on this basis methods can be developed for the conversion of real loads into standard loads.

In such methods the criterion of equivalence can be defined in terms of failure or unserviceability or, simply, in terms of weight. The more elaborate procedures of conversion of real loads into standard load intensities by equalizing the load intensities which produce failure or unserviceability of the structure are applied whenever the differences between the actual load-patterns and the standard pattern produce significant differences between the gross weights of the actual load and of the equivalent standard load as, for instance, in short-span highway bridge girders that can accommodate a single traffic unit only (truck or truck-combination) in critical position. The standard load becomes a function of the span, type and structural action of the designed structure; therefore standard load intensities of similar structure only can be related. As the length of the span increases and more traffic units are accommodated, differences in pattern become less significant and a simple conversion of the actual loads into standard loads of the same gross weight may be sufficiently accurate.

An extensive investigation has recently been published[12] in which the load patterns of several thousand heavy trucks and truck-combinations of 14 different types, established in the 1942 loadometer survey, have been converted into equivalent standard H-truck loads for simple bridge girders of spans between 10 and 100 ft. The basis of comparison was the maximum bending moment produced by the equivalent loads; for the simple span this is identical with a comparison of conditions of either failure or unserviceability. The results of this investigation clearly show the effect of span on the standard load; for all vehicles considered the average equivalent H-truck load intensity increases from, roughly, 12 tons for the 10 ft. span to 18 tons for the 100 ft. span, while the span independent mean gross weight equivalent is as high as 21 tons.

For spans that are long in relation to the length of the traffic unit the actual load pattern is relatively unimportant; a uniform distributed standard load is usually determined from the gross weight of the applied load units and the space which they occupy on the structure under operating conditions[2].

When a load is always applied in essentially the same pattern, the standard load pattern differs very little from the real load. Loads of this type are "proportional" loads by definition, such as wind- and gust-forces, or mass-forces acting at the mass centers of the structures (acceleration and seismic forces, centrifugal forces). The pattern of such loads is usually very simple.

Since the structures for the design of which standard loads are established are expected to serve their purposes during service periods

extending into the future, while the load pattern applied in their design has necessarily been derived from load-observations preceding the time of design, a certain adjustment of the standard load has to be made, to embody anticipated future changes in the past or present loading conditions. Such adjustments are purely matters of engineering judgment, based on a judicious evaluation of the trend in the general operating conditions of the designed type of structure.

6. Random Variation of Standard Load Intensity

The intensity of real loads and of standard loads can usually be considered as a design parameter subject to random variations. Probability concepts and statistical methods constitute, therefore, adequate procedures for prediction and presentation of this variation in a form suitable for use in design, except when a maximum load intensity of relatively high frequency of occurrence can actually be specified in a non-statistical manner, as in the cases of floorloads in warehouses, of fluid-loads and-pressures in storage tanks, and of maximum train-loads on railroad bridges. In all these cases the maximum load is not the result of a random variation, but of arbitrarily established limitations; this is illustrated by the difference in the concept of the maximum loads used in the design of a water storage tank and of a multiple purpose dam. For the storage tank the maximum load, due to a completely filled tank, represents the maximum design load, to be considered in relation to serviceability rather than to failure. The maximum load to be carried by the dam in the courses of its anticipated life time however can only be statistically predicted from extrapolation on the basis of previous records of the random variation of flood-levels; it represents a "catastrophic" load to be sustained without failure, but probably not without a certain damage, which is acceptable because of its very rare occurrence, and because its complete prevention would require the consideration of the rare "catastrophic" load as the actual service load, and might thus lead to an economically unjustifiable increase in the cost of the structure.

When the variation of the load intensity is random, the form of such variation is described either by its frequency distribution which indicates the relative frequencies of occurrence $f_i(S)$ of various load intensities S, or by its cumulative frequency functions $F_i(S) = \sum_0^i f_i(S)$, indicating the relative frequencies of occurrence of load intensities smaller than S, or $F_i^*(S) = \sum_i^\infty f_i(S) = 1 - F_1(S)$, indicating the relative frequencies of occurrence of load intensities equal to or higher than S.

For proportional loads such functions are obtained from systematic and sufficiently extensive records of observed load-intensities by the application of standard statistical methods to the collected data; extensive records of wind-and gust-loads on structures[13], of flood-levels in rivers[19] have, in fact, been so analyzed and presented.

Information concerning the variation of observed load intensities is, however, of limited value, unless it is possible to deduct from it the general pattern of random variation in terms of theoretical probability

distributions p(S) or probability functions

$P(S) = \int_0^S p(S)\, dS$ and $P^*(S) = 1 - P(S)$, respectively. The assumption appears then justified that the random variation of the observed parameter is governed by the same probability rules that produce the functions p(S) and P(S) or $P^*(S)$, and can, therefore, be predicted beyond the range of actual observation by the application of these rules.

In engineering applications it is, in general, more expedient to use the cumulative frequency and probability functions $F_i(S)$ or $F_i^*(S)$ and P(S) or P*(S) respectively, rather than the associated distribution functions $f_i(S)$ or p(S) since even for a limited number n of observations, arranged in ascending order, the ordinates of the cumulative frequency functions $F_m(S) = m/(n+1)$ or $F_m^*(S) = 1 - m/(n+1)$, representing the frequency of occurrence of values smaller or larger than the m - th observation, can be easily determined; on the other hand the direct establishment of the associated distribution function is not possible unless the number of observations is very large. Moreover, the availability of various types of probability paper, on which, as a result of a suitable distortion of the probability scale the respective probability function is transformed into a straight line, makes a simple graphical fitting of the particular P(S) function to the plotted points $F_m(S)$ possible and, in most cases, sufficiently accurate.

If P*(y) denotes the probability of a value larger than y, the reciprocal $N(y) = 1/P^*(y)$ defines the "return-number" of observations, which is the mean number of observations expected to contain one value exceeding y, or the "return period", if the observations are made at fixed time-intervals.

The very nature of design requires extrapolation of the fitted function, since load-intensities that have actually occurred during the necessarily limited period of observation, can usually not be considered rare enough to be associated with failure or, if the period of observation is relatively short, even with unserviceability. Hence, the selection of a probability function to represent the observed frequency function involves more than simple curve fitting; unless the probability function correctly describes the general pattern of variation, extrapolation towards the extremes (tails of the distribution function) will result in erroneous predictions within the extreme range of variation, which is just the range relevant for design. Therefore, distribution functions that have been developed essentially for curve fitting purposes are useless where, as in design, the main purpose of the fitted function is to be a guide for extrapolation from the range of the observation towards the region of small probabilities. Only those functions can fulfill this purpose which represent a definite pattern of variation, resulting from certain basic assumptions concerning its origin.

Both continuous and discrete functions of such general type can be applied to the study of random variations of load intensities, depending on the formulation of the problem and on the kind of information available.

On the basis of records of observation of load intensities, taken at

certain time intervals, it is possible to formulate the problem in alternative ways:

(a) Considering the load intensity as a continuously changing variable, a continuous probability function is fitted to the observed frequency function;
(b) Dividing the range of variation of the load intensity by a suitably chosen limiting (lower) load level the occurrence of intensities exceeding this level are considered as discrete events, and their frequencies fitted by a suitable discrete probability distribution.

When the load intensity cannot be obtained directly by observation, as in the case of highway bridges, it is derived from information concerning the expected proportion of traffic units of various types and weight by combining the standard load units into patterns of various probabilities of occurrence and by computing their respective gross weights.

7. Principal Types of Random Variation of Load

Continuous distribution functions which represent distinct patterns of variation of a dimensionless random variable y are:

(a) the Normal or Gauss-Laplace distribution (Fig. 1)

$$p(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \tag{1}$$

This distribution is symmetrical and expresses the fact that the random variable y is the sum of a large number of independent random variables of arbitrary distribution:

(b) the Logarithmic-Normal distribution (Fig. 2)

$$p(y) = \frac{1}{y\sqrt{2\pi}} \exp \frac{1}{2} (\log^2 y)\, dy \tag{2}$$

The distribution is skew and may be interpreted in two alternative ways: either as the result of the fact that y is the product of a large number of independent variables, or by assuming that y depends on the consecutive operation of a large number of independent variables in such a way, that the change in y is proportional to the momentary value of y, that is to the sum of the effects of all previously operating variables (history-effect).

(c) The distributions of Extreme Values in Groups of n Observations (Fig. 3)

These distributions are obtained from the conditions that in such a group (n-1) values are either smaller or larger than the value of the variable y, so that y represents either the largest or the smallest value of the group. If $\overline{P}(y)$ and $\overline{p}(y)$ denote, respectively, the probability-and the distribution-function of the infinite population of observations, out

of which finite groups of n observations each have been formed, the following expressions are obtained by application of simple rules of combination for the probability that y is, respectively, the largest or the smallest value of the group:

$$p_1(y) = n[1-\bar{P}(y)]^{n-1}\,\bar{p}(y) \tag{3a}$$

and

$$p(y) = n\bar{P}(y)^{n-1}\,\bar{p}(y) \tag{3b}$$

For distributions of the original population $\bar{p}(y)$ which approach zero as a negative power of e (exponential type) and for values of $n \to \infty$, the following so-called "asymptotic" functions, valid for large sample sizes n, have been obtained for, respectively, the distributions of the largest and of the smallest values of such a population[20]:

$$P^*(z) = \exp(-e^{-z}) \qquad p^*(z) = \exp[-z-e^{-z}] \tag{4a}$$

and

$$P_1(z) = 1-\exp(-e^{z}) \qquad p_1(z) = \exp(z-e^{z}) \tag{4b}$$

z stands for a "reduced" variable given by

$$z = \alpha(y-v) \tag{5}$$

where y is the observed variable and v represents the most probable extreme value (mode of the distribution of y) while $1/\alpha$ is a measure of dispersion, proportional to the standard deviation.

Since this type of extreme value distribution is unlimited in both directions, its application to engineering problems, in which only positive values of the variable are physically possible, is open to similar objections as the application of the normal distribution. This objection can be met by a simple logarithmic transformation,

$$y = \ln x, \quad v = \ln u$$

as a result of which Eqs. (4a) and (4b) are transformed into

$$P^*(x) = \exp\left[-\left(\frac{x}{u}\right)^{+\alpha}\right] \qquad p^*(x) = \alpha\left(\frac{x}{u}\right)^{+\alpha-1}\exp\left[-\left(\frac{x}{u}\right)^{+\alpha}\right] \tag{4c}$$

for the largest values, and

$$P_1(x) = 1-\exp\left[-\left(\frac{x}{u}\right)^{\alpha}\right] \qquad p_1(x) = \alpha\left(\frac{x}{u}\right)^{\alpha-1}\exp\left[-\left(\frac{x}{u}\right)^{\alpha}\right] \tag{4d}$$

for the smallest values. These distributions are limited at $x = 0$, being valid, respectively for $x \leqq 0$ and $x \geqq 0$ only.

The logarithmic transformation can be justified either on the basis of expediency, as simply producing an extreme value distribution of the physically desired limitation, or on the basis of the assumption that the variation in the original population is not the result of more or less independent additive effects, but of cumulative effects, leading to a logarithmic normal distribution for the original population.

Distributions of extreme values are of considerable engineering importance. Many statistical problems relevant to design can be formulated in terms of extreme values: only the relatively high out of the infinitely large "population" of load-intensities are representative of design loads. Hence, gusts may be considered as extreme values of randomly varying wind loads, floods as extremes of randomly varying water-levels, high maneuver loads of aircraft as extremes of random operating flight loads. Since for the application of Eqs. (4) the actual shape of the distribution of the original population is irrelevant, as long as it approaches zero exponentially in the direction of either large or small values, a condition that most statistical populations fulfill at least in fair approximation, Eq. (4c) can be expected to represent variations of certain loads sufficiently well to permit extrapolation towards the limiting values of small probability. This has been confirmed by statistical analysis of observations of gust loads, maximum wind velocities, flood levels, and live loads on floors.[6]

Discrete probability distributions are used either for the prediction of the frequency with which the intensity of a proportional load will exceed a limiting intensity within a given number of observations, or for the prediction of the probability of occurrence of combinations of various standard load units in the total number of units making up a complex load. In the first case the average frequency of such an intensity, or its "return number" or "return period," must be obtained from sufficiently extensive load records; in the second case the average proportion of various types of traffic units and the load distribution within each type must be known. If only two types of load units are considered, both types of problem can be dealt with by the Binomial distribution, which is given by the successive terms of the expansion $[p + (1 - p)]^n$, where p is the probability of occurrence of an intensity belonging to one group, or of one type of load-unit, $(1-p)$ the probability of its non-occurrence, or of the occurrence of the alternative load-unit, and n the total number of loading-cycles or of load-units involved. The m - the term of the above expansion

$$p(m) = \frac{m!}{n!(n-m)!} p^m (1-p)^{n-m} \tag{6}$$

gives the probability of m occurrences of the first alternative in n load cycles or load-units. The sums

$P(m) = \sum_{0}^{m-1} p(m)$ or $\sum_{0}^{m} p(m)$ and $\sum_{m}^{n} p(m)$ give respectively, the

probabilities of less than, or not more than $\underline{m}$, and of at least $\underline{m}$ occurrences. The average number of occurrences is $m_o = np$.

In two previous papers[1, 2] frequency distributions of the number of trucks in the total number of vehicles on a bridge have been computed on the basis of the Binomial distribution.

The terms of the Binomial distribution can be replaced by those of the Poisson distribution

$$p(m) = (np)^m \exp(-np)/m! \tag{7a}$$

(which can be evaluated with comparatively little effort, and for which very extensive tables have been computed), when $\underline{n}$ tends towards infinity and $\underline{p}$ towards zero, while the average number of occurrences $m_o = np < 5$. The small values of the "a priori" probabilities $\underline{p}$ characterize the occurrences governed by the Poisson distribution as "rare events."

Conversely, therefore, occurrences which have the character of "rare events" may a priori be assumed to be governed by the Poisson distribution. Since the high load-intensities relevant in design should, by definition, occur infrequently enough to classify them as "rare" or "relatively rare" events, the Poisson distribution is likely to provide a reasonable approach for their prediction. This conclusion has been confirmed, for instance, with respect to flight-loads (multiples xg of the acceleration of gravity g): the frequency of occurrence of the relatively rare, large values of $\underline{x}$ is closely reproduced by a Poisson distribution.[21] Thus, if V – g records give the return period $\underline{T}$ of load intensities equal to or exceeding (xg), the average number of occurrence of such a load during a certain time t is (t/T); this number replaces the number (np) in Eq. (7a). Hence, the probability that a flight-load-intensity equal to or exceeding (xg) may be expected $\underline{m}$ times during an interval t is given by

$$p(m) = (t/T)^m \exp(-t/T)/m! \tag{7b}$$

The probabilities that such a load intensity will occur not more than $\underline{r}$ times or at least $\underline{r}$ times during T are, therefore,

$$P(r) = \exp(-t/T) \sum_{0}^{r} (t/T)^m/m! \tag{8a}$$

and

$$P^*(r) = \exp(-t/T) \sum_{r}^{\infty} (t/T)^m/m! \tag{8b}$$

respectively.

If it can be assumed as, for instance, in fatigue that the (r + 1) -th occurrence of this load-intensity will produce a critical condition (failure) of the considered structure, which has survived $\underline{r}$ repetitions of this load, Eq. (8a) represents the function of survival of the structure. The mortality distribution $p_F(t)$ expressing the probability of failure as a function of service-time $\underline{t}$ is obtained by differentiating [1 - P(r)], or

(9) $$p_F(t) = -dP(r)/dt = (t/T)^r \exp(-t/T)/r!T \qquad \text{for } t>0$$

which is known as the gamma-distribution (or Type III Pearson distribution). It has a maximum at t = rT which, therefore, represents the most probable time to failure, a result that could be expected. The cumulative distribution $P_F = \int_0^t p_F(t)\,dt$, which represents the probability of failure between t = 0 and t is the Incomplete Gamma function, which has been tabulated by Pearson. A continuous distribution function can thus be derived from a problem originally formulated in terms of a discrete distribution.

The Poisson distribution can also be used to predict the distribution of the intensity of a standard load, if the problem is suitably formulated, by specifying a lower limit of this intensity as the "zero-level," and dividing the range of load intensities above the zero-level into equal load-steps. The principal consideration for the specification of the "zero-level" is the condition that the occurrences of load-intensities exceeding this level should be sufficiently "rare" events to be governed by the Poisson distribution; the number of load-steps in excess of the zero-level is then considered the random variable. Hence, the probability of occurrence of a load-intensity exceeding the "zero-level" by m load steps is given by

(7c) $$p(m) = K^m e^{-K}/m!$$

where K is the number of load steps by which the average load intensity exceeds the "zero-level".

It is interesting to note that in the investigation of highway bridge loadings referred to previously[12] Poisson distributions have been compared with frequency distributions of equivalent H-truck standard loads derived, for short spans, from the 1942 loadometer survey. Such comparison of the statistically presented loadometer data and the Poisson distribution, on the basis of plotted cumulative functions, shows that in most cases the agreement is quite disappointing, particularly for the gross-weight distribution. Thus extrapolation on the basis of the Poisson distribution towards high load values would lead to erroneous predictions. This is clearly illustrated for the combined data of all trucks in Fig. 4, in which the frequency functions for the equivalent H-truck loading for 50 ft. and 100 ft. spans and for the gross weight are plotted on logarithmic-normal probability paper, and compared with the respective, continuously plotted Poisson functions.

It appears that in this case the definition of "heavy trucks" has not been narrow enough to justify analysis by the "rare event" approach. Limitation of the analysis to trucks of equivalent H-loads exceeding, for instance, 20 tons might have produced better agreement with the Poisson distribution than has been obtained by including the relatively frequent trucks of lighter weight; it may also be expected that the distribution

of real "rare events", such as strong wind-and gust-loads, seismic forces and wave-forces can be predicted by the Poisson distribution with fair accuracy, if the problems of variation are formulated in terms of discrete probabilities.

According to Fig. 4 the logarithmic normal distributions fit the data much better than Poisson distributions, particularly with respect to possible extrapolation. In the only case for which the Poisson distribution provides a good fit (50 ft. span) the logarithmic-normal and the continuously plotted Poisson distribution cannot be easily distinguished.

When a complex load produces, simultaneously, two independent types of structural action, as in the case of an airplane-wing loaded in bending by forces resulting from acceleration, and in torsion by forces resulting from speed, the information concerning the variability of the loading is represented by a three-dimensional probability surface, instead of a two-dimensional probability distribution. The ordinates of this surface are a measure of the expected frequencies of occurrence of the various loading conditions defined by the points in the reference plane, the coordinates of which represent the intensities of the two component loading actions. Such surfaces can be constructed from V-g records by simply counting the number of occurrences of various combinations in the total number of observations, and tracing contour lines connecting points of equal frequency of occurrence (Fig. 5).[3] Similar problems arise in the design of long-span bridges, particularly suspension bridges, under the joint action of traffic load and wind, as well as in other types of structures and machine parts subject to dual load action.

8. Time-Dependence of Standard Load

Since the properties of the structural materials by which failure and unserviceability of a structure are determined, are not time-independent, the time-dependence of the application of the standard load will significantly affect the critical operating conditions of the structure. The principal aspects of this time-dependence are: the rate of load application, the duration of the loading and the number, frequency and sequence of load repetition.

The rate of load application defines the possible "spectrum of behavior" enclosed by "static" loading on the one side, and by shock-loading or impact on the other. In general, the longer the span and the larger the mass of the structure, the narrower the range of variation in the rate of load-application. In certain, relatively short-span structures as, for instance, in bridge stringers and floor-beams, or in airplane wings, the full range of variation is possible; in main girders of long-span bridges the range of variation is very limited because of the limitation of the variation of the speed of traffic. In the case of floor loads or of snow loads in buildings, or in the other extreme case of operating loads on hammer-foundations or on blast-resisting structures, the possible range of variation of the rate of load application within each of the two structural types itself is negligibly narrow.

The duration of the loading is significant in design only when it is long enough to produce either a significant permanent deformation or a significant reduction in the fracture-strength of the material, or both.

In this case the critical operating conditions of the structure are determined with reference to the end of the expected period of service of the structure, during which the load may have been sustained continually or intermittently for long periods. Loading conditions of this type usually arise in the design of long-span or prestressed concrete structures, or in the design of metal structures or parts for service at elevated temperatures.

The effect of the number of load repetitions on the failure condition of the structure determines their significance in the particular design; only under exceptional conditions does the effect of frequency become important, as in the cases of very low frequency of application of loads to highly time-sensitive materials that may arise in high-temperature design. Since the fracture strength of most engineering materials decreases with increasing number of repetitions of a load intensity exceeding a critical level ("fatigue"), the effect of load repetitions on the failure load of the structure must usually be analyzed when more or less frequent load repetition is an integral part of the operating conditions. A reasonably close estimate of the expected number of applications of various intensities of the standard load, therefore, represents an important part of the load-information, particularly since these figures are also of considerable importance with respect to the specification of an acceptable probability of failure (see Section 11).

While it is known that the sequence in which the load-intensities are applied considerably affects the number of load applications that can be sustained by the structure without damage or failure ("cumulative fatigue damage"), the available information concerning the effect of sequence is very scarce. The statistical distribution of the standard load-intensity determines the type of load-sequences to be expected: statistical time-series can be derived from any particular type of distribution. It is with respect to such random time-series of load-intensity that the fatigue life of a material should be determined. It is only recently that some investigations[22, 23] have been started along these lines: preliminary results have demonstrated the inadequacy of the conventional concept of "cumulative fatigue damage", which defines equality of damage at different load levels by equality of "cycle-ratios" (n_i/N_i) where n_i denotes the number of load-cycles of a certain intensity actually applied, and N_i the number of load-cycles producing fatigue fracture at the same load intensity. However, no generally valid alternative concept has so far been established.

III. Structural Analysis

9. Conditions of Unserviceability and Structural Failure

(a) Unserviceability

Only idealized materials are truly and invariably elastic, plastic or viscous, since this is how they have been defined. The behavior of real materials is a combination of the various components, and changes with the state of stress and its intensity, with the applied stress-rate and the external temperature. Therefore, terms such as "brittle" or

"ductile" can only refer to a particular state of a material, not to the material in general. With change in the loading and external conditions one or the other of the component types of deformation may become dominant, so that the material will appear very nearly elastic under one set of conditions, and highly viscous under another. Thus, the question whether a specific method of inelastic analysis is always applicable to the design of structures of a particular material is meaningless, since different conditions of loading and temperature require different methods of evaluation of the load-intensities leading to unserviceability or failure. Hence, any design procedure in which these two critical limits are correctly related to the applied loading represents a "limit design" procedure.

For the evaluation of the limit of serviceability methods of elastic analysis are usually adequate. Only in the few cases in which localized inelastic strain, produced by the first few applications of the service load not only within the structural connections but in and near the critical sections or members of the structure itself, is considered acceptable, the relation between the intensity of the applied load and the extent of the inelastic deformation must be determined by analysis beyond the elastic range. Such cases will arise when a considerable increase of the load beyond the elastic limit-load is accompanied by a very limited amount of inelastic deformation, and are limited to conditions in which either a very steep elastic stress gradient exists (as in beams of sharp curvature or thick-walled tubes), or the resisting section is of a shape that is marked by a considerable difference between the elastic and the fully plastic moment (as in diamond-shaped or circular beams) and, therefore, by a high so-called plastic "shape factor", or where both conditions occur simultaneously (as in the design of eye-bars, chain-links or cranehooks). "Shape-factors" which specify the ratio between the fully plastic and the elastic moments of the section vary between 1.10 to 1.20 for wide-flanged structural sections, to 2.20 for diamond-shaped sections. Little advantage is therefore gained by permitting a small plastic deformation under service conditions in members made of structural shapes, unless the stress-gradient is very steep, as in corners of frames; large increases of the service loads of crane-hooks, chain-links or thick-walled tubes can, however, be obtained as a result of permitting localized, elastically constrained plastic deformation.

Methods of analysis of structures and structural parts under conditions of elastically constrained plastic deformation are well established for relatively simple structures and shapes (straight beam, circular tube). The design criterion usually adopted is a limiting depth of penetration of the plastic deformation, a limiting amount of overall permanent deformation of the structure, or a limiting strain in the critical section (0.1 to 0.2 percent). Approximate methods for the evaluation of the depth of penetration as a function of the applied load have recently been developed.[24]

For structures of time-sensitive materials, such as concrete, plastics, or metals at elevated temperatures, the application of visco-elastic analysis may become necessary even for the evaluation of the limiting service load, since a maximum permanent deformation at the end

of the expected period of service of the structure is introduced as a design criterion whenever the time-dependent component of the deformation ("creep") cannot be neglected. In present methods of visco-elastic analysis[25] this component is generally assumed to be a linear function of time, and a linear or a power-function of stress. The acceptable design limit of permanent deformation is determined by the maximum change of shape which the structure can suffer without being in any way affected in its proper functioning.

(b) Failure

Elastic analysis is, in general, inadequate for the evaluation of the failure load of metal structures, unless failure occurs by general or local elastic instability of members in compression or in bending, or after a very large number of repetitions of stresses far below the yield-stress ("fatigue"). Elastic buckling and, to a limited extent, fatigue represent the principal conditions for which elastic analysis leads to "limit design".

Most other conditions require the application of methods of "plastic" analysis in the evaluation of the failure load of metal structures. These methods are generally based on the assumption that, after reaching the yield-stress, the metal yields at constant stress (ideal plasticity); they lead to two different types of failure, one characteristic for proportional loads, the other for complex loads. These are referred to in the very extensive literature on the subject, as, respectively, total "plastic collapse", and "alternating plasticity" or "progressive collapse".[9]

Conditions of "plastic collapse" are produced by a proportional increase of the acting load-system until the structure collapses as a kinematical chain (mechanism) after the formation of a sufficient number of "plastic hinges". Methods of plastic collapse analysis determine the sections in which plastic hinges can possibly occur, as well as the alternative arrangements of such hinges which might transform the considered structure into a kinematical chain, together with the respective load-intensities that are necessary to produce the various arrangements. The collapse load is the lowest load-intensity that can transform the structure into a mechanism; the ratio between this load and the limiting elastic load has recently been considered as a "safety factor",[26] though it represents but a plastic load factor.

Collapse analysis is usually not concerned with the order in which plastic hinges occur, nor with the local strain in them, although it is quite probable that, before the last hinge produces the theoretical collapse, some of the previously formed hinges have undergone very large plastic strains which, in real materials, may be associated with strain-hardening, fracture or local instability.

Conditions of "alternating plasticity" or of "progressive collapse" are produced by the repeated application of a complex load system as a result of which one or more sections of a structure in consecutive loading cycles are subject either to plastic strains of opposite direction or to plastic strain in the same direction, whenever a certain loading sequence is applied. It is assumed that the alternating plastic strain gradually produces an "exhaustion" of the cohesion of the metal,

terminating in fracture; progressive plastic strain in the same direction on the other hand is assumed to produce, finally, plastic collapse.

Since alternating plasticity and progressive collapse produce failure after a relatively small number of load applications (it is this fact which distinguishes alternating plasticity from fatigue, where the number of load cycles to failure is large), any sequence which results in either condition is to be considered a failure loading. If, however, under a certain sequence of application of a standard load the alternating or the progressive plastic strain tends towards a limiting value, so that the plastic deformation is stabilized before failure occurs, and the structure responds elastically to any subsequently applied arbitrary sequence of this load, the structure is said to have attained a condition of "shake-down", as a result of which failure by alternating plasticity or progressive collapse has been permanently prevented.[27] The highest intensity of the standard load for which a "shake-down"of the structure is still possible therefore represents the lower limit of the standard failure-load intensity. This limiting "shake-down" load is usually much lower than the plastic collapse load under the same proportionally applied load pattern, sometimes even as low as the elastic limit-load. Most of the excess of the plastic collapse load over the elastic limit-load may, therefore, be lost when the operating conditions of the structure require analysis for repeated sequences of complex loading.

As a result of discrepancies between the assumed and the real response of the structure, the indiscriminate application of plastic analysis for the evaluation of the failure load is, however, limited even in cases where analysis for proportional loading only is required. The full plastic collapse load of a redundant structure depends on the development of the full plastic carrying capacity of all plastic hinges by which the structure is transformed into a mechanism, notwithstanding the large hinge rotations associated with such transformation. The two effects which most seriously interfere with such development of the full plastic hinge moments are: local lateral buckling of flanges,[28] and discontinuous spread of plastic deformation along glide lines, as a result of which plastic wedges are formed in advance of the isotropically spreading plastic regions.[29] Separately or in conjunction, these effects tend to reduce the carrying capacity of the plastic hinges and thus the plastic collapse load of the structure. Tests of mild steel continuous beams have shown that only under exceptionally favorable conditions can the theoretical load be attained; in most cases not more than one-half to two-thirds of the excess of the plastic collapse load over the elastic limit load can actually be realized,[30a] particularly when the loads are applied at very slow rates or sustained over several hours.[30b]

The choice of the appropriate method for the evaluation of the failure load of reinforced concrete structures depends on the percentage of reinforcement in the critical sections, on the type of the structure and on its operating conditions. So-called "plastic" design methods,[31] based on an assumed analogy of behavior with metal structures in bending appear to be useful for weakly reinforced sections, the carrying capacity of which is exceeded by yielding of the reinforcement at relatively low compressive stresses in the concrete, which produces the

effect of a "plastic hinge"; the question whether the triangular, parabolic or any other shape of distribution of the compressive stresses is more adequate has, in this case, very little meaning. Any stress-distribution can be fitted to the test data, provided the ratio between the critical compressive fiber-stress in bending, computed on the basis of the assumed distribution, and the standard uniform cylinder compression strength is accepted as the empirical parameter that it actually is.

It should, however, be realized that the apparent "plasticity" of concrete subject to high compressive stresses is the manifestation of a progressive internal disruption of the material, starting far below the conventional "compressive strength" obtained in the relatively high-speed standard tests, rather than plastic flow of a type occurring in metals. Hence, its effect on the failure load of the structure will be different from that of metal plasticity, particularly for repeated or sustained application of loads producing high stresses. Since practically all test results have been obtained under conditions of short-time "static" tests, the extrapolation of those results to the condition of failure even after a small number of load repetitions does not seem justified. With respect to failure of reinforced concrete after a large number of load repetitions (fatigue), it is generally assumed, though on the basis of insufficient experimental evidence, that a load of roughly one-half of the static failure load can be applied an infinite number of times to a reinforced concrete beam.[12]

In the analysis of failure due to instability (buckling) or due to restraints on free expansion and contraction (temperature-stresses and stresses due to motion of the supports) of structures of reinforced concrete and of other time-sensitive materials the time dependence of the deformation ("creep") or of the stresses ("relaxation") must be considered. Thus, the axial force on a column, or the dead load thrust in a long-span slender concrete arch-rib will produce additional time-dependent bending moments as a result of which the failure load of the strut or rib is reduced.[32] Conversely the internal moment of the prestressing force in a prestressed concrete girder gradually diminishes as a result of the relaxation of this force due to concrete creep, reducing the failure load of the beam.

Stress relaxation also prevents the full build-up of elastic temperature stresses[33] or other restraint-stresses in structures or elements of time-sensitive materials, particularly metals at elevated temperatures.

10. Variation of Unserviceability and Failure Loads

The variation of the critical design loads is essentially the result of the random variations of the pertinent material properties. In redundant structures a further source of variation is the development of alternative mechanisms of transformation into the state of "plastic collapse".

In metal structures the load associated with unserviceability depends mainly on the yield-stress; the failure load depends on either the yield-stress or the fatigue-strength of the material. In concrete structures either the yield-stress of the steel or the compressive strength of the

concrete determine the failure load, while a compressive concrete stress defined by an acceptable amount of inelastic strain, or a tensile stress in the steel producing damaging cracks in the concrete may determine the limiting condition of unserviceability.

The random variation of the yield-stress, of the fatigue strength and of the compressive strength of concrete has been investigated in recent years on an expanding scale. The character of the frequency distributions of these properties has by now been reasonably well established.

The approach to the analysis of random variation of the relevant strength properties is similar to that applied in the analysis of the variation of load intensities. Since, for design purposes, predictions must be made of the probability of occurrence of strength values that are smaller than those that have actually occurred in the performed tests, extrapolation of the fitted probability function towards improbably low strength values becomes necessary. As in the case of loads, such extrapolation can be made with some confidence only if the fitted probability function represents the general pattern of variation that can reasonably be expected.

It is not unreasonable to assume that only the lowest values of the results obtained in tests of samples of a certain number of material specimens are of significance in the prediction of the low range of failure loads on which a rational estimate of the probability of failure must necessarily be based. Hence, extrapolation of strength values should be based on the limited probability function of smallest values, Eq. (4d), fitted to the test results that have been arranged and plotted in ascending order.

Attempts to fit extreme value distributions of type (4d) to strength data have, in general, shown that a good fit can be obtained, particularly over the range of low strength values. While in the vicinity of the mean or the mode no differentiation between various possible distributions is possible, the range of high strength values is better fitted by the distribution function of the population itself than by that of its extreme values.[6]

Instead of using two different distribution functions for low and high strength values of the same test series a reasonably good fit of strength data can usually be obtained by using the logarithmic-normal distribution. The advantage of using the logarithmic-normal distribution is that the statistical procedures developed for the normal distribution, particularly the methods for the interpretation of the results of small numbers of tests, can be applied to the interpretation of strength data and to their extrapolation towards the range of the small values pertinent in design.

Fig. 6 represents the results of, respectively, 3125 and 850 mill tests of the yield-stress of ASTM-A7 structural steel[14] plotted on logarithmic-normal probability paper with straight lines graphically fitted to the data. This presentation shows that the probability of occurrence of values lower than the specified minimum (33,000 psi) is at least 5×10^{-3}. Considering that the results of mill tests are consistently higher than those of standard tests as a result of the higher testing speeds, it appears that the specification of a minimum of 27-28,000 psi

might be more reasonable, if the probability of values falling below the minimum is to be kept lower than 10^{-3}.

Fig. 7 represents the result of 721 tests of the yield-stress of eye-bars,[14] also plotted on logarithmic-normal probability paper and fitted by a straight line. In this case the probability of occurrence of values smaller than the specified minimum is as high as 0.12.

Fig. 8 shows the results of 673 compression tests of structural concrete manufactured under reasonably good control,[15a] fitted by a logarithmic-normal distribution. Because of the fact that, in spite of a mean of 5320 psi, the minimum has been specified as low as 3825 psi, the probability of values lower than the minimum could be kept down to about 10^{-3}.

It appears that inadequate control produces a distribution that is closer to an extreme value distribution over its whole range than to a logarithmic normal distribution, as indicated by Fig. 9, in which the results of 296 compression tests of concrete manufactured under difficult control conditions,[15a] have been plotted on extremal probability paper and fitted by a straight line representing Eq. (4d). The conclusion to be drawn from this diagram is that inadequate control reduces the extremes, rather than the mean value, increasing the range and number of low test results and sharply reducing the number of high values.

The principal sources of non-random variation of the strength properties that determine unserviceability or failure loads are the variations of the applied strain- or stress-rate and the effect of size on the strength properties. Thus, for instance, the large difference in the loading rates of floor beams and of main girders in a long-span steel bridge may produce significant differences in the stresses at which the first yielding occurs in the two structural parts. Considering, for instance, that on a 300 foot span highway bridge with floor beams spaced at 20 ft. the critical floor-beam load (single or double truck axle) might travel at 50 or 60 mph, while the critical girder load (sequence of vehicles of high load intensity) travels at about 12 mph,[2] the ratio of the loading rates of the floor beams and the girders may be as high as 75, or almost two orders of magnitude. Such differences in loading rate produce quite significant differences in yield-stresses; for mild steel a yield-stress of 30,000 psi obtained under the standard ASTM test strain rate of about 10^{-4} in/in/sec. is increased by about 20 percent at a strain rate of 10^{-2} in/in/sec. and by about 50 percent at a strain rate of 1 in/in/sec., produced by a blast-load of 0.001 sec. duration.

It is also quite well known that the yield-stress of steel obtained in the relatively high-speed mill tests is substantially (10 to 15 percent) higher than that obtained in standard tests or in low-speed structural loading. Moreover, it is common knowledge that the yield-stress decreases with the size of the section or its diameter (in reinforcing bars), and even for the same structural section is considerably higher for specimens taken from the web than for those taken from the flanges; differences of 15 percent are not unusual.

These influences have to be considered in conjunction with the random variations, but cannot be combined with them; rather, their effect

should be considered by shifting the mean of the distribution of the random variation in accordance with the expected trend of the non-random variation.

IV. Probability of Failure

11. Choice of Decision Rule

The acceptable probability of failure or of unserviceability can be specified either arbitrarily, or in relation to the expected number of load applications, or on the basis of an economic balance between the cost of increasing the safety and the cost of failure. The choice of the type of procedure or "rule" used for the specification of the probability of failure depends on the importance and cost of the structure, as well as on the consequences and cost of failure. It is evident that the probability of failure of an important structure or of a structure the failure of which would endanger human life in peace time should be practically zero. It can, however, never be theoretically zero, unless the design could be based on an absolute upper limit of the load-intensity, and on a distribution of the significant strength parameter that is known to be definitely limited at a minimum value; structures designed on such a basis would, however, be inefficient and uneconomical. It is, moreover, not quite logical even to attempt to design for zero probability of failure or unserviceability, since other risks to a structure, such as fire or earthquake are accepted as inevitable. And while all measures are taken to reduce their incidence or the damage associated with their occurrence, the risk remains real, however small, and its magnitude finds its numerical expression in terms of the respective insurance rates, or of that portion of the rates that represent the pure risk. Hence, comparison of the risk of failure or unserviceability with other risks of similar consequences may provide a first rough rule for the specification of an acceptable probability of failure or of unserviceability.

A more rational rule can be derived from the requirement that such a combination of an improbably high load intensity with an improbably low carrying capacity (failure load) which would produce failure of the structure should not be expected to occur during the lifetime of the structure. Hence, the probability of its occurrence should be extremely small or, in other words, the "return period" of such a combination should be very much longer than the lifetime of the structure, of its "return number" much higher than the expected total number of load applications.

In the relatively simple case of the design of aircraft wings, for instance, an estimate of a reasonable ratio between the "return period" of a "failure combination" of applied load and carrying capacity, and the service life of the aircraft may be deduced from flight-insurance rates. Considering, for instance, that transatlantic flight insurance rates are roughly 0.02 percent for about 20 hours flight duration, of which about 1/3 to 1/2 may represent actual risk, and assuming a service life of 10,000 flying hours, the accident risk implied in these figures is two in 10^7 flying hours, or a "return period" of failure of 500 times the expected service life of the aircraft; the accepted chance of an accident is, therefore, 1/500.

In the case of a railroad or highway bridge the total number of standard load applications $\underline{N}$ for the various structural parts can be roughly estimated from the expected total traffic volume per year and the anticipated service life of the structure. The ratio of $\underline{N}$ to the "return number" (inverse probability of occurrence) of "failure combinations" of applied load intensity and carrying capacity expresses the accepted risk of failure.

The economic approach to the specification of an acceptable failure risk[34] is based on the assumption that the cost of failure or of unserviceability can be considered as a charge against the structure, equal to the capitalized total cost of failure $\underline{C}_F$, or of unserviceability $\underline{C}_S$, multiplied by the probability $\underline{P}_F$ or $\underline{P}_S$ of its occurrence. Since the cost of failure and, to a lesser extent, the cost of unserviceability, are necessarily functions of the probability of failure or unserviceability for which the structure has been designed, the optimal economic probabilities of failure or of unserviceability should make the sum of the initial cost of the structure $\underline{A}$ and the capitalized cost of failure or of unserviceability a minimum. Hence, the condition for the optimal probabilities

$$A + P_S C_S \rightarrow \min \quad \text{or} \quad A + P_F C_F \rightarrow \min$$

and

$$(10) \qquad \frac{dA}{dP_S} + P_S \frac{dC_S}{dP_S} + C_S = 0 \qquad \frac{dA}{dP_F} + P_F \frac{dC_F}{dP_F} + C_F = 0$$

The relation between initial cost of the structure and the design probability of failure may, in first approximation, be introduced in the form

$$(11) \qquad A = A_0(1 - c \log P_F) \qquad P_F < 1$$

only the cost of the load-carrying part of the structure will be affected by changes of P. For simple reinforced concrete floors, for instance, a value of $c \doteq 0.018$ has been established from comparative designs.[15b]

The total cost of failure is made up of two components: one which is independent of P_F and includes all direct and indirect losses resulting from the failure, as well as the cost of repair (if such repair is possible), the other that represents the cost of reconstruction when repair is not possible, and depends on P_F in a similar way as the initial cost. Hence, considering that for very long service life $Q \doteq \frac{\gamma}{\gamma - 1}$ represents the factor of capitalization, where γ is the interest factor $(1 + i)$ and $\underline{i}$ the interest rate,

$$(12) \qquad C_F = [C_{OF} + m A_0 (1 - c \log P_F)] Q$$

where $\underline{m}$ denotes the ratio between cost of reconstruction and initial cost; when the structure can be repaired without total reconstruction $C_F = Q\, C_{FO}$.

The cost of unserviceability can, in first approximation, be regarded as independent of P_S; hence, $C_S = QC_{SO}$.

Introducing Eqs. (11) and (12) into the second Eq. (10) the following relation for the evaluation of P_F is obtained.

$$(13) \quad d(\log P_F)\, cA_O(1+mP_F Q)+cmA_O Q \log P_F=(C_{FO}+mA_O)Q$$

Since for any reasonably small probability of failure $mP_F Q$ can be neglected in comparison to one,

$$(14) \quad \frac{1}{P_F}=\left(\frac{C_{FO}+mA_O}{cA_O}-m\log P_F\right)\frac{Q}{\log \underline{e}} \doteq \frac{C_{FO}+mA_O}{cA_O}\frac{Q}{\log \underline{e}}$$

where $\underline{e}$ is the base of the natural logarithms. For instance with $\underline{C}_{FO} = \underline{10}A_O$, $\underline{m} = 2$, $\underline{c} = 0.015$, $\underline{i} = 3\%$, the optimal economical probability of failure $P_F \doteq 1/6300$. Similarly, the relation for the evaluation of P_S is obtained from the first of Eqs. (10), if it is assumed that the relation between A and P_S is of the same type as Eq. (11):

$$(15) \quad \frac{1}{P_S}=C_{SO}\frac{Q}{A_O c}\log \underline{e}$$

Since, in general, the cost of unserviceability C_S will be lower than A_O (unless such condition can only be repaired with considerable difficulty, in which case it should be considered as failure rather than unserviceability and, therefore, $C_S = C_F$) the economically optimal probability of unserviceability $P_S > P_F$. For instance with a high value of $C_S = 0.2A_O$, $c = 0.015$, $i = 3\%$ the design probability for unserviceability $P_S \doteq 1/1050$.

Thus, the economically optimal risk of failure or unserviceability, for which the structure should be designed, decreases with increasing ratio of cost of such failure or unserviceability to the initial cost, as well as with increasing service life and decreasing interest rate.

A reasonable estimate of the acceptable risk of "failure"- or "unserviceability combinations" of applied load intensity and carrying capacity can now be obtained by combining the economically optimal risk of failure with the expected total number $\underline{N}$ of applications of the standard load, in the form (P_F/N) and $P_S/N)$ respectively. The number $\underline{N}$ to be used depends on the distribution of standard load intensities on which the design of the structure is based. For instance, if the standard load distribution for a highway bridge represents only heavy trucks of gross weight exceeding 20,000 lbs., their number $\underline{N}$ will be a small fraction of the total number of trucks, which has to be considered as $\underline{N}$ if the standard load distribution represents all truck-weights that might be encountered.

12. Analysis of Failure and Survival: Safety Factors

The probabilities of failure or of survival for which structures are designed must be referred to critical operating conditions of the structures; different values of the probabilities are necessarily associated with different operating conditions. Methods for the selection of such probabilities have been discussed in Section 11.

The simplest operating condition of a structure is defined as the joint action of dead load and service load; the action of dead load alone is not an operating condition, since no important structure is built for the purpose of carrying its own weight. The load-intensity $\underline{S}$ associated with this operating condition is the sum of the dead load $\underline{S}_W$ and the live load intensity $\underline{S}_P$; the variation of $\underline{S}$ is therefore made up of the variation of the load components $\underline{S}_W$ and $\underline{S}_P$. Since variations of the dead load are essentially due to unavoidable errors and inaccuracies in design and construction,(1) the distribution of $\underline{S}_W$ can generally be assumed as Normal; the distribution of $\underline{S}_P$, on the other hand, will usually not be Normal. Both distributions must be combined in order to obtain the distribution of their sum $\underline{S}$; this distribution will, therefore, not be of a simple type. Since, in general, the range of variation of the dead load is much narrower than that of the service load, it may be assumed, in first approximation, that the character of the distribution function of $\underline{S}$ is identical with that of $\underline{S}_P$, and that only the values of the mean and the variance are combined by superposition of $\underline{S}_W$ and $\underline{S}_P$. Hence, if the distribution of $\underline{S}_P$ is extremal (unlimited) with mean $\underline{S}_{PO}$ and variance $\sigma_P^2 = \frac{\pi^2}{6}\,\frac{1}{\alpha_P^2}$, while the distribution of $\underline{S}_W$ is normal with mean $\underline{S}_{WO}$ and variance σ_W^2, and if $[1 - (\frac{\sigma_W}{\sigma_P})^2]$ does not appreciably differ from one, the distribution of $\underline{S}$ remains approximately extremal with

$$(16) \qquad S_O = S_{OW} + S_{OP} \qquad \frac{1}{\alpha_S^2} = \frac{6}{\pi^2}\sigma_W^2 + \frac{1}{\alpha_P^2} = \frac{6}{\pi^2}\sigma_S^2$$

The addition of a dead load of comparatively high intensity and low variability to the service load reduces the range of relative variation of the total load intensity $\underline{S}$; the reduction is the higher the larger the ratio (S_{OW}/S_{OP}).

When other load types are included in the operating conditions as, for instance, wind-loads acting jointly with dead load and service load, the load intensity $\underline{S}$ is obtained by combining the intensities and the variations of the three load types. As a result of such combinations the total load intensity $\underline{S}$ is increased while its probable number of applications is reduced. Hence, the design for such operating conditions can be based on a smaller number $\underline{N}$ and thus on a higher probability of failure or unserviceability. In the extreme case of operating conditions defined by the occurrence of destructive forces (earthquake or blast) in addition to the other design loads the expected number of occurrence of such conditions may be as low as 1 or 2; the ratio of the design probabilities of

failure for such extreme load intensities and for the normal operating conditions may, therefore, be as high as $\underline{N}$ for normal operating conditions. Since the failure loads associated with probabilities of failure differing by a factor of $\underline{N}$, that is by several orders of magnitude, may differ quite considerably, relatively high intensities of very infrequent destructive loads usually require not more than a disproportionately small increase of the carrying capacity of the structure to ensure its survival, without however ensuring its complete serviceability.

If $\underline{S}$ denotes the total intensity of the applied loads under specified operating conditions and $\underline{R}$ the intensity of the same type of applied total load that would produce failure of the structure or unserviceability, the relation,

$$r = R - S \leqslant 0 \tag{17}$$

determines the conditions of failure or of unserviceability; the inequality $r > 0$ is the condition of survival. If a "factor of ignorance" $\underline{n}$ is introduced Eq. (17) becomes $r = R - n\,S \leqq 0$, where $n > 1$.

Since the statistical variations of $\underline{R}$ and $\underline{S}$ are independent, the probability of failure $\underline{P}_F$ is the product of the probability $\underline{P}_S$ of occurrence of a load intensity S and the probability $\underline{P}_R$ of occurrence of a failure load $R \leqslant S$. Conversely, the probability of survival $P_F^* = 1 - P_F$ is the product of the probability $\underline{P}_S^*$ of occurrence of a load intensity $< S$ and the probability P_R^* of occurrence of a failure load $\underline{R} > \underline{S}$. Both probabilities can be defined either with respect to failure proper, or with respect to unserviceability, provided the "failure load" $\underline{R}$ is related to actual failure or to unserviceability of the structure, respectively.

When $\underline{S}$ and $\underline{R}$ are normally distributed with means $\underline{S}_O$ and $\underline{R}_O$ and standard deviations σ_S and σ_R, and if two variables $\underline{x}$, $\underline{y}$ are defined by $x = (S - S_O)/\sigma_S = \Delta S/\sigma_S$ and $y = (R_O - R)/\sigma_R = \dfrac{\Delta R}{\sigma_R}$, the distribution functions of $\underline{x}$ and $\underline{y}$ are

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right] \qquad p(y) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}y^2\right] \tag{18}$$

Hence, the probability of any combination of values (x,y), defining a certain value of the variable $\underline{r} = \underline{R} - \underline{S} = (R_O - S_O) - (x\,\sigma_S + y\,\sigma_R)$ is given by

$$p(x) \cdot p(y) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x^2 + y^2)\right] \tag{19}$$

The lines of constant probability $p(x,y) = p_O$ in the coordinate system (x,y) are, therefore, the circles

(20) $$x^2+y^2=-2\ln(2\pi p_o)=-4.606\log(2\pi p_o)$$

centered on $(R_o/\sigma_R, S_o/\sigma_S)$, with radii $\rho = 2.15\sqrt{-\log 2\pi p_o}$. In the same coordinate system the condition r = 0 is represented by the equation of the straight line

(21) $$x\sigma_S+y\sigma_R=R_o-S_o$$

which cuts off the length $(R_o - S_o)/\sigma_S$ on the x-axis and $(R_o - S_o)/\sigma_R$ on the y-axis. Its slope against the x-axis is $\tan\alpha = (-\sigma_S/\sigma_R)$ (Fig. 10). The condition that this line be a tangent to a circle of constant probability defined by Eq. (20):

(22) $$R_o-S_o/\sigma_R=\rho/\cos(\pi-\alpha)=\rho/\sqrt{1+(\sigma_S/\sigma_R)^2}$$

determines the relation

(21a) $$x\sigma_S+y\sigma_R=\rho\cdot\sigma_R\cdot\sqrt{1+(\sigma_S/\sigma_R)^2}$$

which delimits the region of "failure" ($r < 0$) from the region of "survival" ($r > 0$). On this line the failure-combination with the highest probability of occurrence, which is equal to p_o, is defined by the coordinates of the point of contact with the circle:

(23) $$x_o=\rho/\sqrt{1+(\sigma_R/\sigma_S)^2} \qquad y_o=\rho/\sqrt{1+(\sigma_S/\sigma_R)^2}$$

so that $(S - S_o) = x_o\,\sigma_S$ and $(R_o - R) = y_o\,\sigma_R$.

The probability of failure P_F for all combinations ($\underline{S}$, $\underline{R}$) along the line r = O is given by that part of the volume under the surface of rotation p(x,y) which is cut off by a plane along the line r = 0 perpendicular to the (xy) - plane. Since the critical failure combination $(x_o y_o)$ has the highest probability of occurrence along the line r = 0, it represents the combination to be used in design. In a hypothetical design based on the mean values $S = S_o = R = R_o$, for which the line r = 0 according to Eq. (21) degenerates into an arbitrary straight line of negative slope passing through the origin, the chances of failure and survival would be even and equal to $P_F = 1 - P_F = 0.5$, as in this case the plane along r = 0 bisects the volume of p(x,y).

The difference $(R_o - S_o)$ represents the "margin of safety", the ratio R_o/S_o the "factor of safety" with respect to the mean values of

of the applied and the failure load, associated with the design for the critical "failure combination" $\underline{S} = S_O + x_O \sigma_S$ and $\underline{R} = R_O - y_O \sigma_R$. If S_O, σ_S, σ_R and p_O are given, the required mean value R_O and the associated safety factor $\nu = R_O/S_O$ can be computed from Eq. (22) ("design problem"). Conversely, if either R_O, σ_S, σ_R, p_O or R_O, S_O, σ_S, σ_R are given, the permissible mean value of S_O or the probability of failure can be computed ("rating problem").

For instance, with $\sigma_S = 0.2\ S_O$, $\sigma_R = 0.1\ R_O$, $p_O = 10^{-6}$ (so that according to tables of the Normal functions the probability of failure for the critical combination $\underline{P}_F < 0.3 \times 10^{-6}$, being the area of the Normal curve beyond the ordinate p_O) the radius $\rho = 4.91$. Hence, from Eq. (22)

$$(R_O - S_O) / \sqrt{1 + 4\,(S_O/R_O)^2} = 0.49\ R_O$$

from which $S_O = 0.38\ R_O$; this represents a safety factor with respect to the means of $\nu = 2.63$. With $x_O = 2.97$ and $y_O = 3.91$, the critical limit design combination $\underline{S} = 1.60\ S_O = 0.61\ R_O = R$. Increasing p_O by two orders of magnitude to $p_O = 10^{-4}$, so that $P_F < 0.3 \times 10^{-4}$, reduces the radius to $\rho = 3.85$. Hence

$$(R_O - S_O) / \sqrt{1 + 4\,(S_O/R_O)^2} = 0.385\ R_O$$

from which $S_O = 0.47\ R_O$; this is equivalent to a decrease of the required safety factor with respect to the means to $\nu = 2.12$. The critical limit design combination $S = 1.46\ S_O = 0.695\ R_O = R$. Thus, under the specified conditions, an increase of the design probability of failure by two orders of magnitude for the same mean carrying capacity is associated with an increase of the mean (permissible) applied load of roughly 25 percent; conversely, for a specified mean applied load, the required mean carrying capacity of the structure can be reduced by 20 percent.

Eq. (22) can be rewritten so as to permit direct evaluation of the safety factor ν by solving the equation:

$$\text{(22a)} \qquad \nu^2 \left[1 - \left(\frac{\rho \sigma_R}{R_O}\right)^2\right] - 2\nu + \left[1 - \left(\frac{\rho \sigma_S}{S_O}\right)^2\right] = 0$$

The same procedure of safety analysis can be applied if the distribution functions of $\underline{S}$ and of $\underline{R}$ are Logarithmic-Normal, with standard deviations of the logarithms of the variables δ_S and δ_R, provided the variables $\underline{x}$ and $\underline{y}$ are defined by

$$x = (\log S - \log S_O) / \delta_S, \quad y = (\log R_O - \log R) / \delta_R$$

Eqs. (19) and (20) remain valid, while Eq. (21), based on R - S = 0 or log (R/S) = 0, is transformed into

$$\delta_S \cdot x + \delta_R \cdot y = \log \frac{R_o}{S_o} \tag{24}$$

which is a straight line with a slope ($-\delta_S/\delta_R$) against the x-axis, cutting off the length $1/\delta_S \log (R_o/S_o)$ on the x-axis, and $1/\delta_R \log (R_o/S_o)$ on the y-axis. The condition that this line be a tangent to the circle $p(x,y) = p_o$ with radius ρ given by Eq. (20):

$$\frac{1}{\delta_S} \log\left(\frac{R_o}{S_o}\right) = \frac{\rho}{\sin(\pi-\alpha)} = \rho\left(\frac{\delta_R}{\delta_S}\right)\sqrt{1+\left(\frac{\delta_S}{\delta_R}\right)^2} \tag{25}$$

again determines the straight line

$$\delta_S x + \delta_R y = \rho \delta_R \sqrt{1+\left(\frac{\delta_S}{\delta_R}\right)^2} \tag{24a}$$

which delimits the regions of "failure" and of "survival". The failure combination with the highest probability of occurrence is, therefore, determined by Eqs. (23) in which σ_S and σ_R are replaced by δ_S and δ_R. Hence, $\log (S/S_o) = x_o \delta_S$ and $\log (R_o/R) = y_o \delta_R$. For given values of S_o, δ_S, δ_R and p_o the required mean value R_o can be obtained by solving Eq. (25), and vice versa (Fig. 11).

For an extreme value distribution of $\underline{S}$ and $\underline{R}$ [unlimited distribution, Eqs. (4a,b)] or of log S and log R [limited distribution, Eqs. (4c,d)] the probability of failure $\underline{P}_F$ for any combination $(\underline{S}, \underline{R})$ where $S > S_o$, $R < R_o$ can be analytically expressed directly by the product of the probability of occurrence $\underline{P}_S$ of values $\geqslant S$ and the probability of occurrence P_R of values $\leqslant R$, while the probability of survival P_F^* under the same conditions is given by the product of the probability of occurrence P_S^* of values $< S$ and the probability of occurrence P_R^* of values $> R$. This is simpler than for a Normal or Logarithmic Normal distribution where the evaluation of P_F requires the computation of the volume of p(x,y) cut-off by a plane along r = 0.

Using P_F^* in preference to P_F for reasons of expediency, so as to have a 3-dimensional coordinate system (x,y,P_F^*) in which all coordinates increase from the origin, and introducing the reduced variables $x = \alpha_S (S - S_o')$ and $y = \alpha_R (R_o' - R)$ where S_o' and R_o' denote the modes of the distributions and $1/\alpha_S^2 = 6\sigma_S^2/\pi^2$ and $1/\alpha_R^2 = 6\sigma_R^2/\pi^2$

the squares of the measure of dispersion, the probability of survival for unlimited extremal distributions

(26) $$P_F^* = P_S^* \cdot P_R^* = \exp[-(e^{-x} + e^{-y})]$$

The lines of constant probability of survival P_{FO}^*, therefore, are given by the equation

(27) $$(e^{-x} + e^{-y}) = -\ln P_{FO}^* = -2.303 \log P_{FO}^*$$

The point of contact of the tangent

(28) $$r = 0 \quad \frac{x}{\alpha_S} + \frac{y}{\alpha_R} = R_O' - S_O'$$

to any of those curves defined by different values of P_{FO}^* determines the critical failure condition (lowest probability of survival) along this tangent which delimits the region of probabilities of survival $P_F^* > P_{FO}^*$ from that for which $P_F^* < P_{FO}^*$. The intercepts with the x- and y-axes determines again the values of S_O' and R_O' compatible with given values of α_S, α_R and P_{FO}^*, and thus the safety factor with respect to the modes of the distributions.

For extremal distributions limited to positive values of the variables $x = \alpha_S \log (S/S_O')$ and $y = \alpha_R \log (R_O'/R)$, the probability of survival

(29) $$P_F^* = P_S^* \cdot P_R^* = \exp[-(x^{-\alpha_S} + y^{-\alpha_R})]$$

Therefore, the lines of constant probability of survival

(30) $$x^{-\alpha_S} + y^{-\alpha_R} = -2.303 \log P_{FO}^*$$

Again the point of contact of the tangent

(31) $$r = 0 \quad \frac{x}{\alpha_S} + \frac{y}{\alpha_R} = \log\left(\frac{R_O'}{S_O'}\right)$$

to any of the curves $P_F^* = P_{FO}^*$ determines the critical failure condition along this tangent, and its intercepts with the coordinate-axes the safety factor with respect to the modes of the distribution.

It should be noted, though, that even when cumulative functions are used P_F^* is not the true probability of survival since, for reasons of expediency, it refers only to the "quadrant of probable failure" (see Fig. 10) in which $S > S_O$ and $R < R_O$. Failure, however, is also

possible in the "quadrants of improbable failure" where $S < S_o$ and $R < R_o$ or $S > S_o$ and $R > R_o$, provided that in either case $S > R$. Hence, the true probability of survival is equal to the sum of the probabilities of survival in the three quadrants, which is higher than P_F^* computed for the "quadrant of probable failure" alone. In the limiting case of design for S_o' and R_o', for which the line $r = 0$ passes through the origin, the true probability of failure is therefore twice P_F^*; for all other conditions it is larger than P_F^* and smaller than $2P_F^*$, so that P_F^* is a safe estimate of the true probability of survival.

When a logarithmic scale for P_F^* is used in a three-dimensional coordinate system, the "surface of survival" Eq. (26) slopes down exponentially in the directions of increasing x and y measured in a natural scale. When x and y are also plotted on a logarithmic scale the "surface of survival" Eq. (29) slopes down in the direction on increasing x and y according to $x^{-\alpha_S}$ and $y^{-\alpha_R}$. The use of the extreme value probability scale for P_F^*, as a result of which the functions $P_S^*(x)$ and $P_R^*(y)$ are transformed into straight lines, transforms both surfaces into hyperbolic paraboloids with straight line generatrices in the directions of the coordinate axes (Fig. 12).

When the distributions of S and R cannot be analytically expressed but only plotted, a purely graphical procedure of safety analysis may be used in which the product $P_F^* = P_S^* P_R^*$ is computed for a sufficient number of points in the (x,y) -plane to make the drawing of the contour lines $P_F^* = P_{FO}^*$ by interpolation possible. The tangents to these lines inclined towards the x-axis by $(-\sigma_S / \sigma_R)$ define the critical failure combinations for limit design, as well as the factors of safety with respect to the mean for various probabilities of survival.

CONCLUSIONS

In the analysis of a structure the analyses of the applied loads as well as of the probability of failure or unserviceability are as necessary and important steps as the analysis of the stresses. The information required for such an integrated analysis which produces a rational figure for the safety of the structure related to acceptable probabilities of failure are the distribution functions of the intensity of an applied standard load, and of the standard load-intensity producing, respectively, structural failure and unserviceability, both presented in a form which justifies extrapolation towards high values of the applied load and low values of the failure- and unserviceability-loads. Since structural failure is usually associated or preceded by substantial inelastic deformation, inelastic analysis of the carrying capacity should replace the elastic stress analysis in the computation of the probability of failure, while elastic stress analysis is in most cases sufficient to determine the probability of unserviceability.

The existing data suggest that either the Logarithmic-Normal or the Extreme Value distribution functions represent actual load- and strength-distributions fairly well, so that they can be used in the analysis of the probability of failure and of the safety of structures. For either type of distribution the safety factors associated with specified probabilities of failure or of unserviceability can be rapidly computed or easily determined by a graphical procedure.

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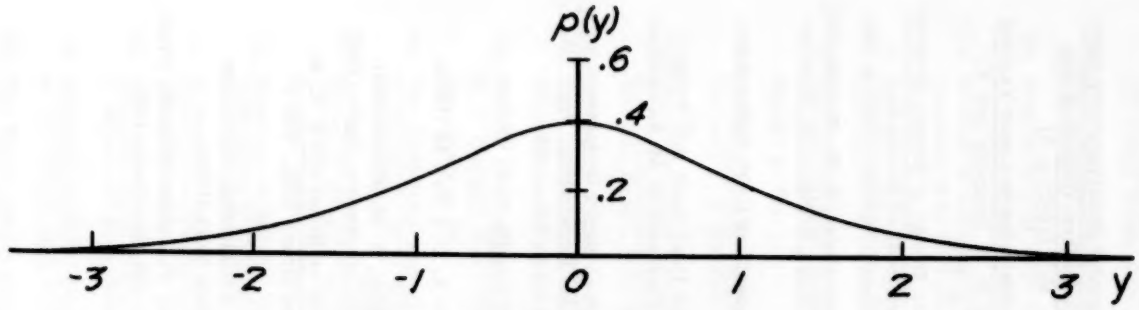

Fig. 1 Normal distribution function.

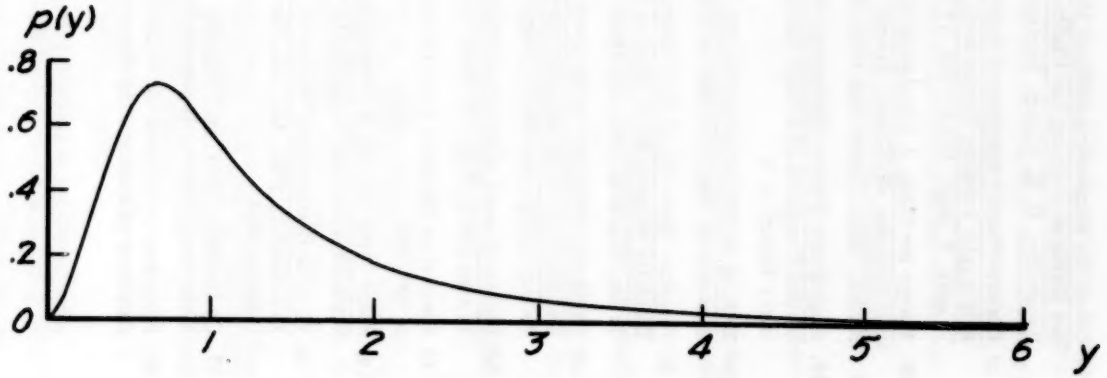

Fig. 2 Logarithmic Normal distribution function.

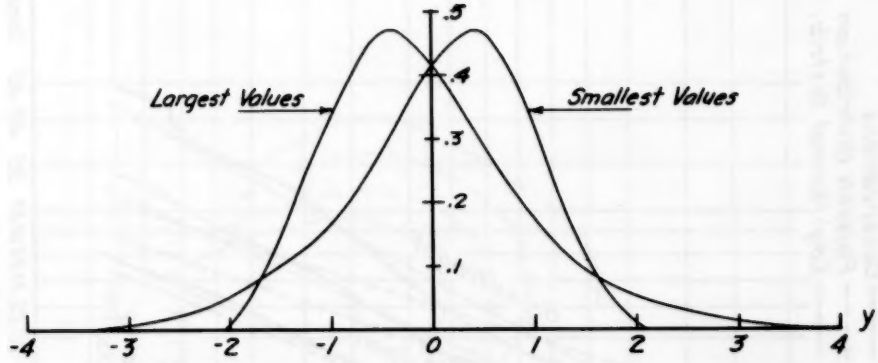

Fig. 3a Unlimited distribution functions of largest and smallest values.

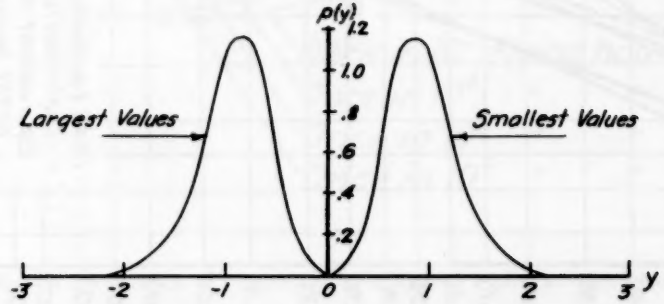

Fig. 3b Limited distribution functions of largest and smallest values.

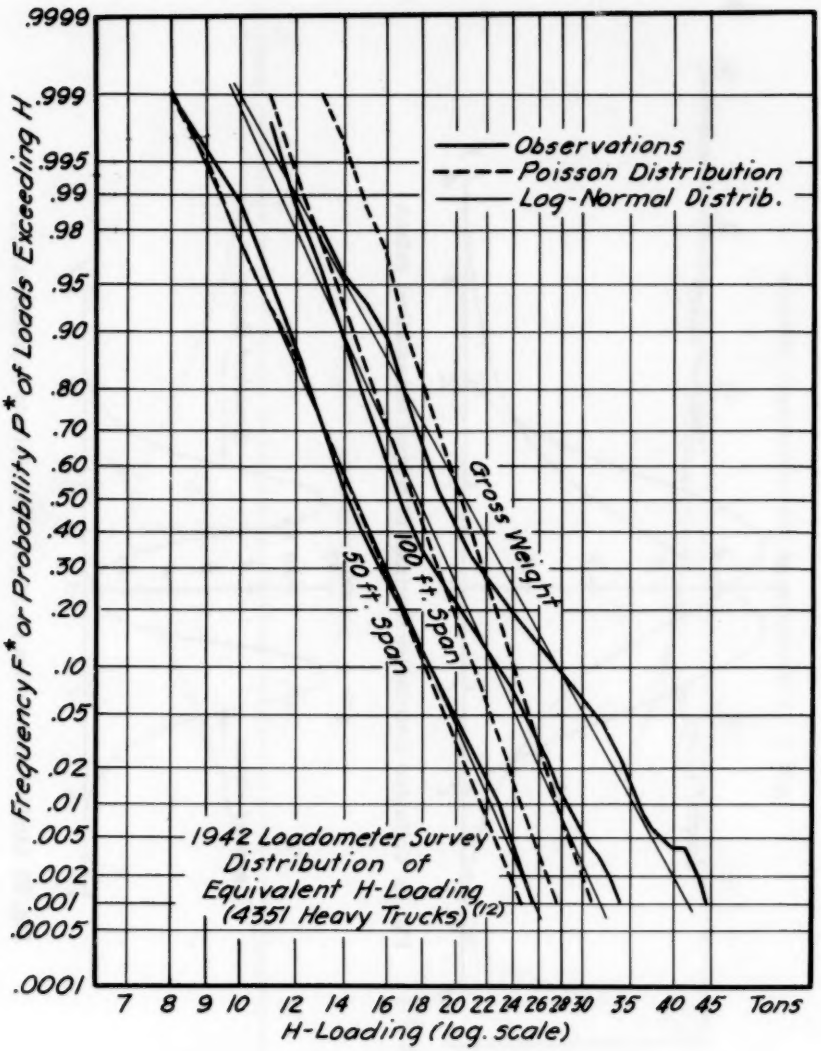

Fig. 4 Comparison of results of 1942 loadometer survey of heavy trucks[12] with Poisson and Log-Normal distributions.

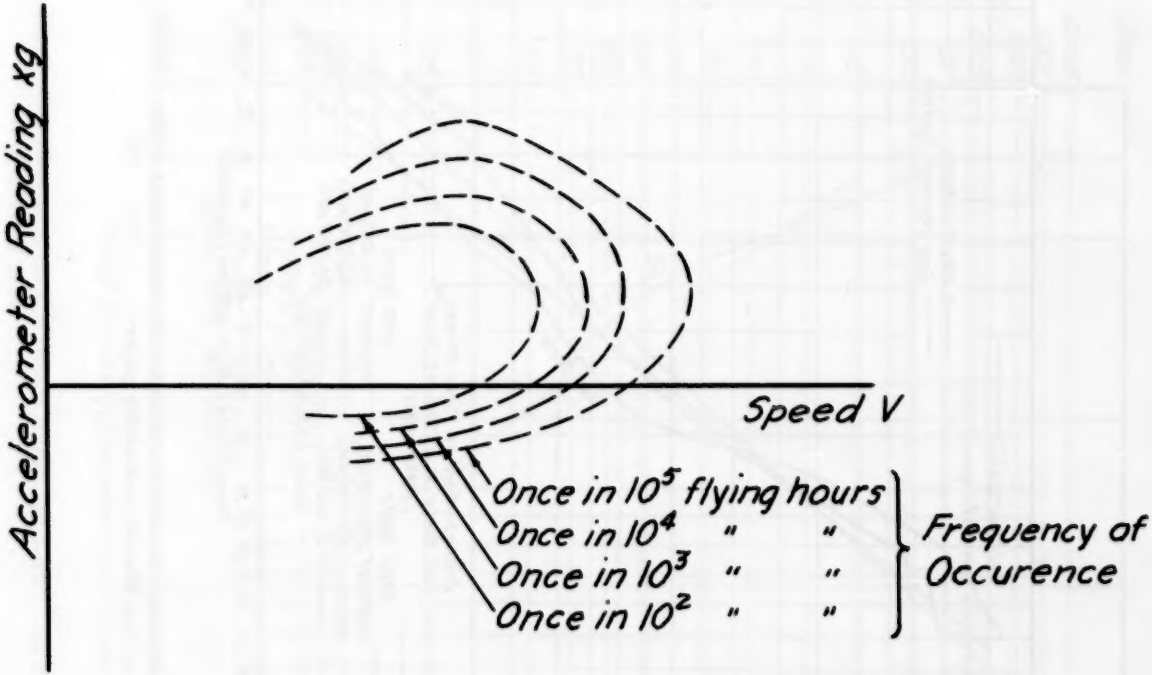

Fig. 5 Probabilities of occurrence of combinations of Speed (V) and acceleration (xg) of aircraft in service.[3]

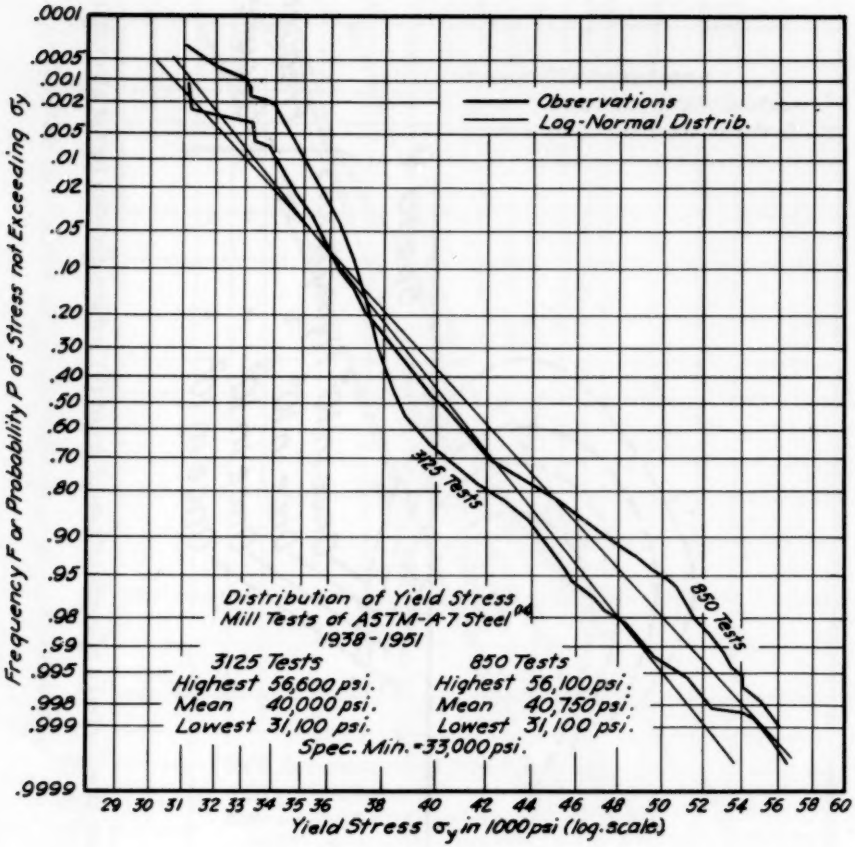

Fig. 6 Distribution of yield stress for structural steel ASTM-A-7[14] on Log-Normal probability paper.

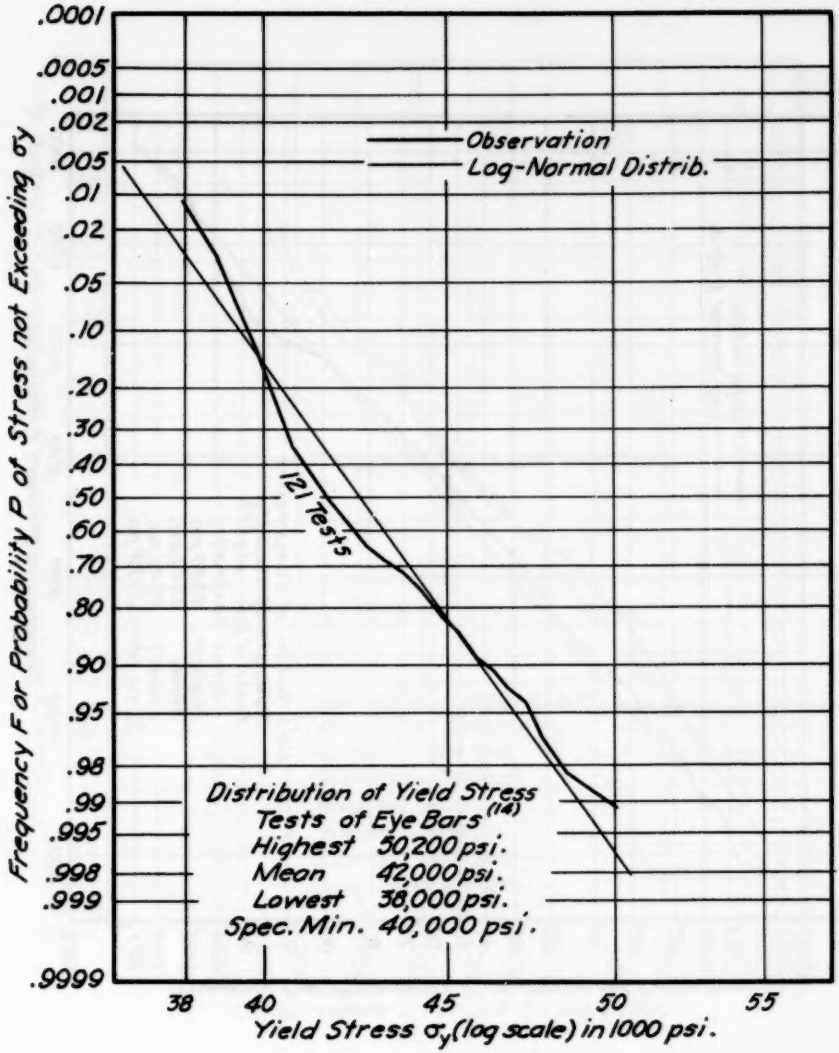

Fig. 7 Distribution of yield stress of bridge eye-bars[14] on Log-Normal probability paper.

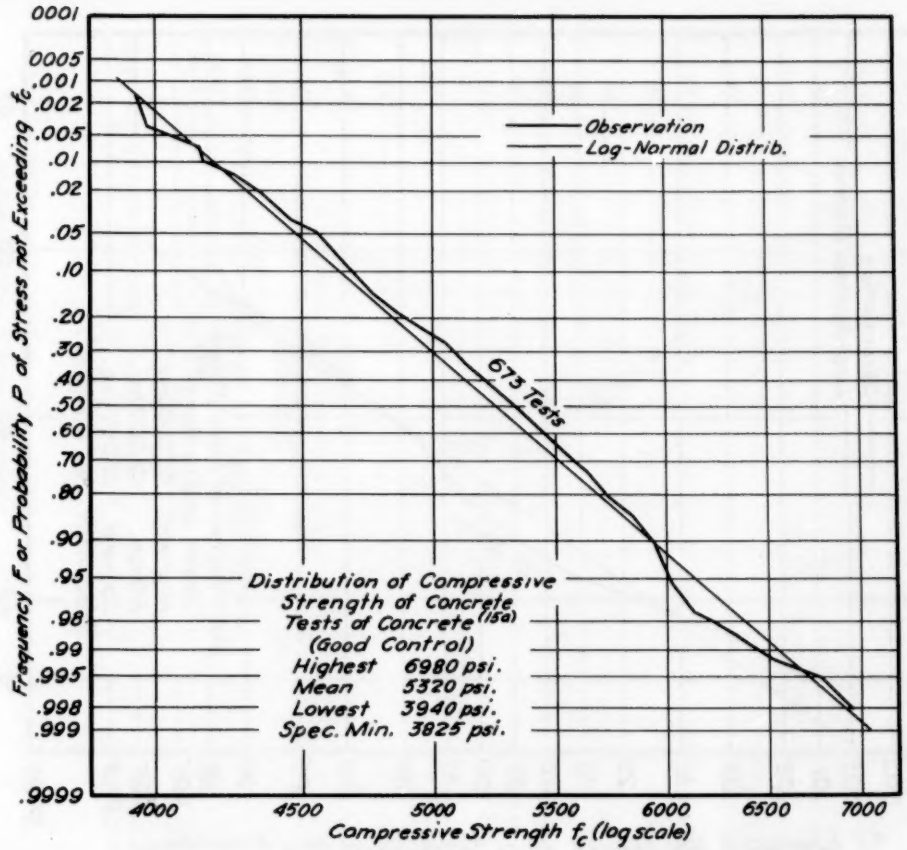

Fig. 8 Distribution of compressive strength of concrete under good control[15a] on Log-Normal probability paper.

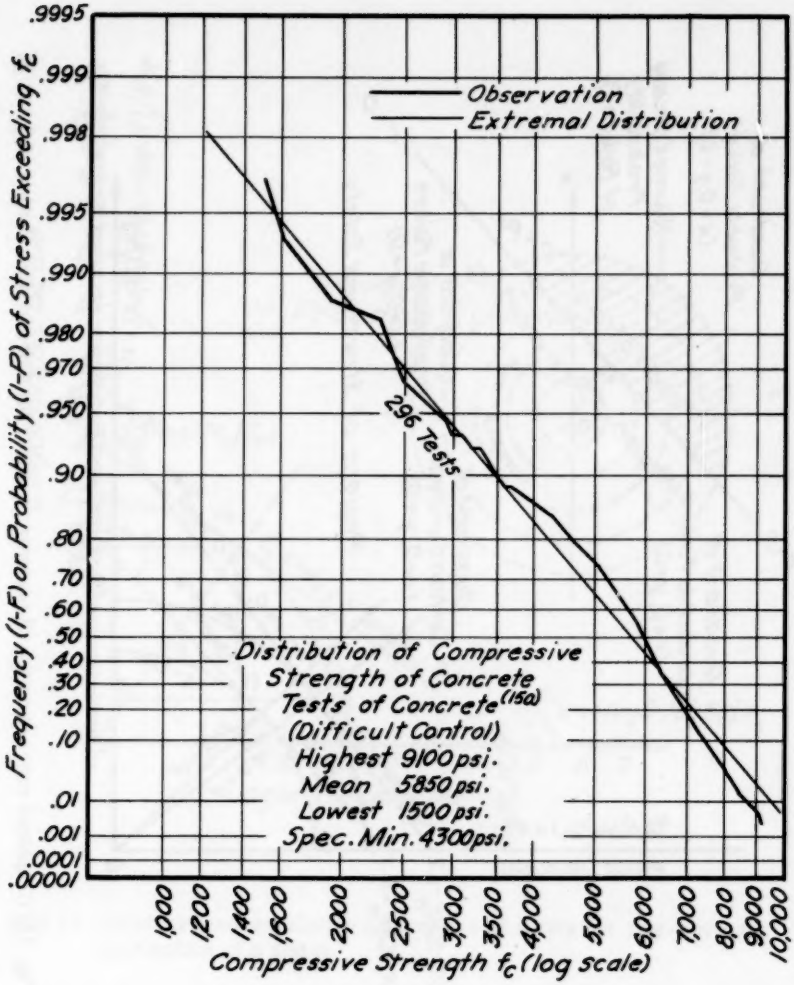

Fig. 9 Distribution of compressive strength of concrete under difficult conditions of control[15a] on Extremal probability paper.

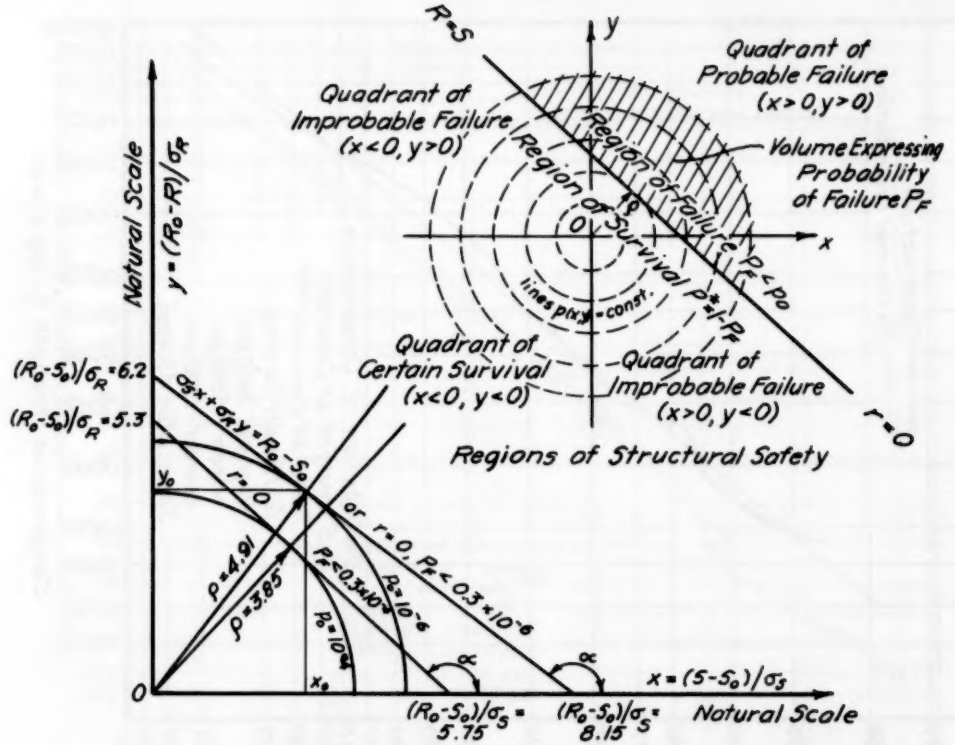

Fig. 10 Graphical representation of analysis of safety for Normal distribution of S and R.

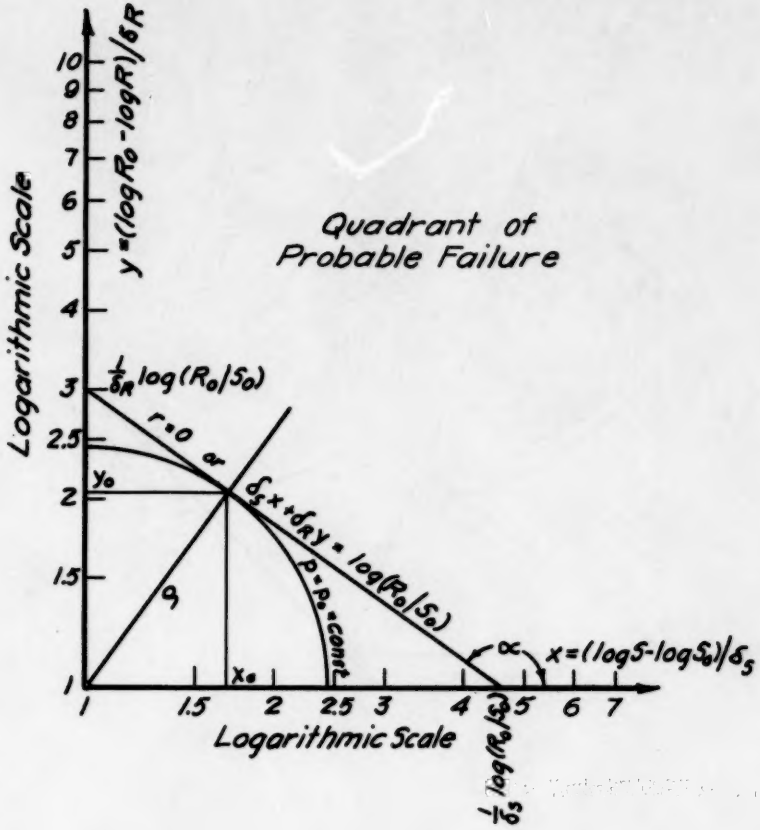

Fig. 11 Graphical representation of analysis of safety for Log-Normal distribution of S and R.

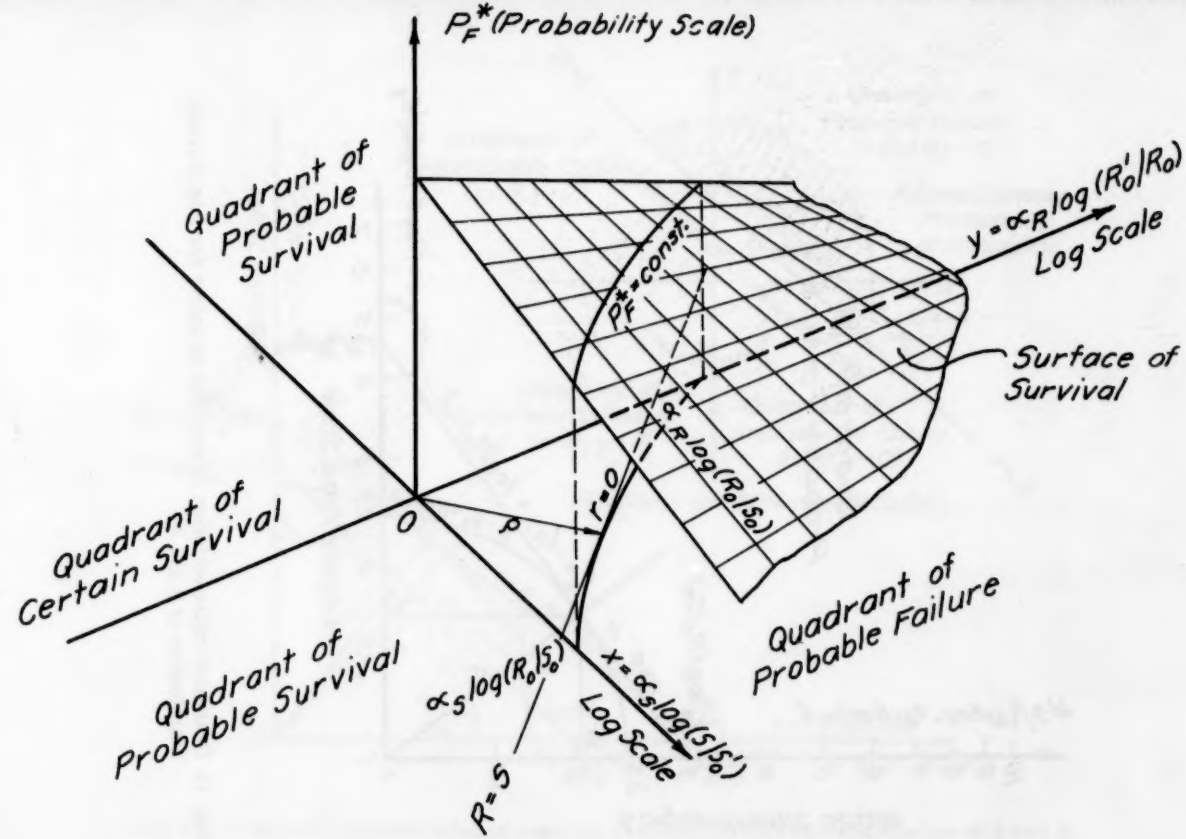

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Fig. 12 Hyperbolic paraboloid surface of probability of survival for logarithmic extremal distributions of S and R using extreme value probability ordinates.